

FACTORIZATION, MAJORIZATION, AND DOMINATION FOR LINEAR RELATIONS

SEPPO HASSI AND HENK DE SNOO

To our friend Zoltán Sebestyén on the occasion of his 70th birthday

ABSTRACT. Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces. Let A be a linear relation from \mathfrak{H} to \mathfrak{H}_A and let B be a linear relation from \mathfrak{H} to \mathfrak{H}_B . If there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that $ZB \subset A$, then B is said to dominate A . This notion plays a major role in the theory of Lebesgue type decompositions of linear relations and operators. There is a strong connection to the majorization and factorization in the well-known lemma of Douglas, when put in the context of linear relations. In this note some aspects of the lemma of Douglas are discussed in the context of linear relations and the connections with the notion of domination will be treated.

1. INTRODUCTION

Let A and B be a pair of linear relations with their domains of definition in the same Hilbert space \mathfrak{H} and their ranges in the Hilbert spaces \mathfrak{H}_A and \mathfrak{H}_B , respectively. The relation B is said to dominate the relation A if there exists a bounded linear operator Z from \mathfrak{H}_B to \mathfrak{H}_A such that $ZB \subset A$. Domination is preserved when the closures of A and B are considered. In the particular case that A and B are, not necessarily densely defined, operators this is equivalent to $\text{dom } A \subset \text{dom } B$ and the existence of a constant $c \geq 0$ such that $\|Af\| \leq c\|Bf\|$ holds for all $f \in \text{dom } A$. The notion of domination, which is familiar from measure theory, plays an important role in the theory of Lebesgue type decompositions. This notion and its role in Lebesgue type decompositions for a pair of bounded operators go back to Ando [1]; it has a similar position when decomposing a nonnegative form with respect to another nonnegative form, see [11], or when decomposing an unbounded operator or a linear relation [12, 13, 14], where some further history and references can be found.

In the present paper it will be shown that domination is closely related to the following well-known lemma of R.G. Douglas [6] when that lemma is put in the context of unbounded linear operators or, more generally, linear relations.

Lemma 1.1 (Douglas). *Let $A, B \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$, the bounded everywhere defined linear operators from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} . Then the following statements are equivalent:*

- (i) $\text{ran } A \subset \text{ran } B$;
- (ii) $A = BW$ for some bounded linear operator $W \in \mathbf{B}(\mathfrak{H})$;
- (iii) $AA^* \leq \lambda BB^*$ for some $\lambda \geq 0$.

If the equivalent conditions (i) – (iii) hold, then there is a unique operator W such that

- (a) $\|W\|^2 = \inf\{\mu : AA^* \leq \mu BB^*\};$
- (b) $\ker A = \ker W;$
- (c) $\operatorname{ran} W \subset \overline{\operatorname{ran}} B^*.$

In the literature one can find a statement which is equivalent to the three items in Lemma 1.1, namely

- (iv) $AA^* = BMB^*$, where $M \in \mathbf{B}(\mathfrak{H})$ is nonnegative and $\|M\| \leq \lambda$.

One may take $\operatorname{ran} M \subset \overline{\operatorname{ran}} B^*$. In addition to the results in the above lemma Douglas indicated some further results for the case when A and B are densely defined closed linear operators; see [6]. Various extensions of these basic results by Douglas can be found in the literature; see, for instance, [4, 7, 8]. The factorization aspect of the Douglas lemma was recently put in the context of linear relations by D. Popovici and Z. Sebestyén [16]; see also some refinements by A. Sandovici and Z. Sebestyén [17]. For the majorization aspect of the Douglas lemma, see [3].

The contents of the present paper are now briefly explained. For closed linear operators or relations A and B the following equivalence will be established in Theorem 3.4:

$$A \subset BW \quad \Leftrightarrow \quad AA^* \leq c^2 BB^*,$$

where W is a bounded linear operator and $c \geq 0$, in fact $\|W\| \leq c$. This result characterizes majorization in terms of a simple factorization type inclusion. Domination for a pair of closed linear operators or relations can be characterized in a similar way:

$$ZB \subset A \quad \Leftrightarrow \quad A^*A \leq c^2 B^*B,$$

see Theorem 4.4. Some consequences of these results will be explored in Section 3 and Section 4. In particular, a characterization of the equalities $A = BW$ and $ZB = A$ is given. For bounded linear operators the factorization $A = BW$ in the original Douglas lemma can be directly connected to the notion of domination for linear relations by means of the following observation:

$$A = BW \quad \Leftrightarrow \quad WA^{-1} \subset B^{-1},$$

see Lemma 5.1. This last equivalence, when combined with the two earlier equivalences, provides a simple proof for the characterization of the ordering of non-negative selfadjoint relations in terms of resolvents; see Theorem 5.2. For the convenience of the reader some results concerning closed nonnegative forms and associated linear relations will be recalled in Section 2.

2. PRELIMINARIES

Let H be a linear relation from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} ; i.e., H is a linear subspace of the product $\mathfrak{H} \times \mathfrak{K}$. The domain, range, kernel, and multivalued part of H are denoted by $\operatorname{dom} H$, $\operatorname{ran} H$, $\ker H$, and $\operatorname{mul} H$. The formal inverse H^{-1} of H is a relation from \mathfrak{K} to \mathfrak{H} , defined by $H^{-1} = \{\{f', f\} : \{f, f'\} \in H\}$, so that $\operatorname{dom} H^{-1} = \operatorname{ran} H$, $\operatorname{ran} H^{-1} = \operatorname{dom} H$, $\ker H^{-1} = \operatorname{mul} H$, and $\operatorname{mul} H^{-1} = \ker H$. For $\mathcal{L} \subset \mathfrak{H}$ the set $H(\mathcal{L})$ is a subset of \mathfrak{K} defined by

$$H(\mathcal{L}) = \{h' : \{h, h'\} \in H \text{ for some } h \in \mathcal{L}\}.$$

In particular, $H(\{0\}) = \operatorname{mul} H$.

Let H_1 and H_2 be relations from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} . Then H_1 is a *restriction* of H_2 and H_2 is an *extension* of H_1 if $H_1 \subset H_2$.

Proposition 2.1. *Let H_1 and H_2 be relations from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} and assume that $H_1 \subset H_2$. Then the following statements are equivalent:*

- (i) $\text{dom } H_1 = \text{dom } H_2$;
- (ii) $H_2 = H_1 \hat{+} (\{0\} \times \text{mul } H_2)$.

Moreover, the following statements are equivalent:

- (iii) $\text{ran } H_1 = \text{ran } H_2$;
- (iv) $H_2 = H_1 \hat{+} (\ker H_2 \times \{0\})$.

Proof. By symmetry it suffices to show the equivalence between (i) and (ii).

(i) \Rightarrow (ii) It suffices to show that $H_2 \subset H_1 \hat{+} (\{0\} \times \text{mul } H_2)$. Let $\{h, h'\} \in H_2$. Since $h \in \text{dom } H_2 \subset \text{dom } H_1$, there exists an element $k' \in \mathfrak{K}$ such that $\{h, k'\} \in H_1$. Hence, with $\varphi' = h' - k'$, it follows that

$$\{h, h'\} = \{h, k'\} + \{0, \varphi'\},$$

and thus $\{0, \varphi'\} \in H_2$ or $\varphi' \in \text{mul } H_2$. Hence (ii) follows.

(ii) \Rightarrow (i) This implication is trivial. \square

The useful result in the following corollary can be found in [2].

Corollary 2.2. *Let H_1 and H_2 be relations from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} and assume that $H_1 \subset H_2$. Then the following statements are equivalent:*

- (i) $H_1 = H_2$;
- (ii) $\text{dom } H_1 = \text{dom } H_2$ and $\text{mul } H_1 = \text{mul } H_2$;
- (iii) $\text{ran } H_1 = \text{ran } H_2$ and $\ker H_1 = \ker H_2$.

Corollary 2.3. *Let H_1 and H_2 be relations from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} and assume that $H_1 \subset H_2$. Then*

- (i) $\text{dom } H_1 = \mathfrak{H}, \text{mul } H_2 = \{0\} \Rightarrow H_1 = H_2$;
- (ii) $\text{ran } H_1 = \mathfrak{K}, \ker H_2 = \{0\} \Rightarrow H_1 = H_2$.

The sum of two linear relations H_1 and H_2 from \mathfrak{H} to \mathfrak{K} is a linear relation defined by

$$H_1 + H_2 = \{\{f, f' + f''\} : \{f, f'\} \in H_1, \{f, f''\} \in H_2\},$$

while their componentwise sum is a linear relation defined by

$$H_1 \hat{+} H_2 = \{\{f + g, f' + g'\} : \{f, f'\} \in H_1, \{g, g'\} \in H_2\}.$$

Let H_1 be a relation from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{M} and let H_2 be a relation from a Hilbert space \mathfrak{M} to a Hilbert space \mathfrak{K} . The product $H_2 H_1$ is a linear relation from \mathfrak{H} to \mathfrak{K} defined by

$$(2.1) \quad H_2 H_1 = \{\{f, f'\} : \{f, \varphi\} \in H_1, \{\varphi, f'\} \in H_2 \text{ for some } \varphi \in \mathfrak{M}\}.$$

Observe, that

$$(2.2) \quad \ker (H_2 H_1) = H_1^{-1}(\ker H_2) = \{f \in \mathfrak{H} : \{f, \varphi\} \in H_1 \text{ for some } \varphi \in \ker H_2\},$$

and

$$(2.3) \quad \text{mul } H_2 H_1 = H_2(\text{mul } H_1) = \{f' \in \mathfrak{K} : \{\varphi, f'\} \in H_2 \text{ for some } \varphi \in \text{mul } H_1\}.$$

In particular, $\ker H_1 \subset \ker H_1 H_2$ and $\text{mul } H_2 \subset \text{mul } H_2 H_1$. The following identities are also easy to check:

$$(2.4) \quad H H^{-1} = I_{\text{ran } H} \hat{+} (\{0\} \times \text{mul } H) \quad \text{and} \quad H^{-1} H = I_{\text{dom } H} \hat{+} (\{0\} \times \ker H)$$

with both sums direct. Hence, in particular,

$$(2.5) \quad \text{mul } H = \{0\} \Rightarrow HH^{-1} = I_{\text{ran } H}; \quad \ker H = \{0\} \Rightarrow H^{-1}H = I_{\text{dom } H}.$$

The closure of a linear relation H from \mathfrak{H} to \mathfrak{K} is the closure of the linear subspace in $\mathfrak{H} \times \mathfrak{K}$, when the product is provided with the product topology. The closure of an operator need not be an operator; if it is then one speaks of a closable operator. The relation H is called closed when it is closed as a subspace of $\mathfrak{H} \times \mathfrak{K}$. In this case both $\ker H \subset \mathfrak{H}$ and $\text{mul } H \subset \mathfrak{K}$ are closed subspaces.

Let H be a closed linear relation from \mathfrak{H} to \mathfrak{K} . Then $H_{\text{mul}} = \{0\} \times \text{mul } H$ is a closed linear relation and $H_s = H \hat{\ominus} H_{\text{mul}}$, so that $\text{dom } H_s = \text{dom } H$ is dense in $\overline{\text{dom } H} = \mathfrak{H} \ominus \text{mul } H^*$, while $\text{ran } H_s \subset \overline{\text{dom } H^*} = \mathfrak{K} \ominus \text{mul } H$. The operator part H_s and H_{mul} lead to the componentwise orthogonal decomposition

$$(2.6) \quad H = H_s \hat{\oplus} H_{\text{mul}}.$$

The adjoint relation H^* from \mathfrak{K} to \mathfrak{H} is defined by $H^* = JH^\perp = (JH)^\perp$, where $J\{f, f'\} = \{f', -f\}$. The adjoint is automatically a closed linear relation and the closure of H is given by H^{**} . The operator part $(H^*)_s$ is densely defined in $\overline{\text{dom } H^*} = \mathfrak{H} \ominus \text{mul } H^{**}$ and maps into $\overline{\text{dom } H} = \overline{\text{dom } H^{**}} = \mathfrak{H} \ominus \text{mul } H^*$. When H is closed the operator parts H_s and $(H^*)_s$ are connected by

$$(2.7) \quad (H_s)^\times = (H^*)_s,$$

where $(H_s)^\times$ denotes the adjoint of H_s in the sense of the smaller spaces $\overline{\text{dom } H}$ and $\overline{\text{dom } H^*}$.

Let H_1 be a relation from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{M} and let H_2 be a relation from a Hilbert space \mathfrak{M} to a Hilbert space \mathfrak{K} . The product satisfies

$$(2.8) \quad H_1^* H_2^* \subset (H_2 H_1)^*.$$

Moreover, if $H_2 \in \mathbf{B}(\mathfrak{M}, \mathfrak{K})$ then there is actually equality

$$(2.9) \quad H_1^* H_2^* = (H_2 H_1)^*,$$

see [10, Lemma 2.4], so that, in particular

$$H_2 H_1^{**} \subset (H_2 H_1)^{**}.$$

Assume that the relations H_1 and H_2 are closed. In general the product $H_2 H_1$ is not closed. However, if for instance $H_1 \in \mathbf{B}(\mathfrak{H}, \mathfrak{M})$, then the product $H_2 H_1$ is closed.

A linear relation H in a Hilbert space \mathfrak{H} is symmetric if $H \subset H^*$ and selfadjoint if $\overline{H} = H^*$. If the relation H is selfadjoint then H_s is a selfadjoint operator in $\overline{\text{dom } H} = \mathfrak{H} \ominus \text{mul } H$. A linear relation H in a Hilbert space \mathfrak{H} is nonnegative if $\langle h', h \rangle \geq 0$ for all $\{h, h'\} \in H$. In particular a nonnegative relation is symmetric.

An important special case of a nonnegative selfadjoint relation appears when one considers relations of the form T^*T where T is a closed linear relation from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} ; cf. [18].

Lemma 2.4. *Let T be a closed relation from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} . Then the product T^*T is a nonnegative selfadjoint relation in \mathfrak{H} . Furthermore,*

$$(2.10) \quad T^*T = T^*T_s = (T_s)^*T_s,$$

so that in particular

$$(2.11) \quad \ker (T^*T) = \ker T = \ker T_s, \quad \text{mul } (T^*T) = \text{mul } T^* = \text{mul } (T_s)^*.$$

The operator part of T^*T can be rewritten as

$$(2.12) \quad (T^*T)_s = (T^*)_s T_s = (T_s)^\times T_s.$$

Proof. It is clear from the definition that T^*T is a nonnegative selfadjoint relation in \mathfrak{H} . In fact T^*T is selfadjoint since $\text{ran}(T^*T + I) = \mathfrak{H}$, which follows from $\mathfrak{H}^2 = T \hat{\oplus} T^\perp = T \hat{\oplus} JT^*$.

Next, let P be the orthogonal projection from \mathfrak{K} onto $\overline{\text{dom } T^*}$, so that $PT = T_s \subset T$. Since $\text{dom } T^* \subset \overline{\text{dom } T^*} = \text{ran } P$ it also follows that $T^* \subset T^*P$. Hence

$$(2.13) \quad T^*T \subset T^*PT \subset T^*T,$$

so that the inclusions are both equalities. From $PT = T_s$ one obtains that $(T_s)^* = (PT)^* = T^*P$, so that (2.13) leads to (2.10). Since T_s is an operator, (2.11) is immediate from (2.10). Furthermore, (2.12) is clear from (2.7). \square

Lemma 2.5. *Let H be a nonnegative selfadjoint relation in a Hilbert space \mathfrak{H} . Then there exists a unique nonnegative selfadjoint relation K in \mathfrak{H} , denoted by $K = H^{\frac{1}{2}}$, such that $K^2 = H$. Moreover, $H^{\frac{1}{2}}$ has the representation*

$$(2.14) \quad H^{\frac{1}{2}} = H_s^{\frac{1}{2}} \hat{\oplus} H_{\text{mul}}.$$

Proof. It is clear that K defined by the right hand side of (2.14) is a nonnegative selfadjoint relation with $\text{mul } K = \text{mul } H$. To see that $K^2 = H$, let $\{f, f'\} \in K^2$. Then $\{f, \varphi\} \in K$ and $\{\varphi, f'\} \in K$. Clearly,

$$\varphi = H_s^{\frac{1}{2}} f + \alpha, \quad f' = H_s^{\frac{1}{2}} \varphi + \beta,$$

with $\alpha \in \text{mul } H$ and $\beta \in \text{mul } H$. Since $\varphi \in \text{dom } H_s^{\frac{1}{2}}$ it follows that $\alpha = 0$ and $f' = H_s f + \beta$, so that $\{f, f'\} \in H$. It follows that $K^2 \subset H$, and since $K^2 = K^*K$ is selfadjoint, it follows that $K^2 = H$.

In order to show uniqueness, let K be a nonnegative selfadjoint relation such that $K^2 = H$. Then

$$\text{mul } K = \text{mul } H.$$

To see this let $\{0, \psi\} \in K$, then clearly $\{0, \psi\} \in K^2 = H$, so that $\text{mul } K \subset \text{mul } H$. For the reverse inclusion, let $\{0, \psi\} \in H = K^2$. Then $\{0, \varphi\} \in K$ and $\{\varphi, \psi\} \in K$. Since K is selfadjoint it follows that $\varphi = 0$, so that $\{0, \psi\} \in K$ and $\text{mul } H \subset \text{mul } K$. This implies that $K = K_s \oplus H_{\text{mul}}$, where K_s is a nonnegative selfadjoint operator. It will now be shown that $H_s = (K_s)^2$, and since the square root of a nonnegative selfadjoint operator is uniquely determined it follows that $K_s = H_s^{1/2}$.

To see that $H_s = K_s^2$, let $\{f, f'\} \in H_s$. Then $\{f, f'\} \in H = K^2$ and $f' \perp \text{mul } H$. Now $\{f, \varphi\} \in K$ and $\{\varphi, f'\} \in K$ for some $\varphi \in \overline{\text{dom } K} = \overline{\text{dom } H}$. Hence $\{f, \varphi\} \in K_s$ and $\{\varphi, f'\} \in K_s$, so that $\{f, f'\} \in K_s^2$. Thus $H_s \subset (K_s)^2$. For the converse inclusion, let $\{f, f'\} \in (K_s)^2$. Then $\{f, \varphi\} \in K_s \subset K$, $\{\varphi, f'\} \in K_s \subset K$, so that $\{f, f'\} \in K^2 = H$. Since $f' \perp \text{mul } H$, it follows that $\{f, f'\} \in H_s$. \square

Let H be a nonnegative selfadjoint relation. Since Lemma 2.5 implies that $\text{mul } H^{\frac{1}{2}} = \text{mul } H$, it follows that

$$(H^{\frac{1}{2}})_s = (H_s)^{\frac{1}{2}},$$

so that the notation $H_s^{\frac{1}{2}}$ is unambiguous. Furthermore it is clear that

$$(2.15) \quad \text{dom } H \subset \text{dom } H^{\frac{1}{2}} \subset \overline{\text{dom } H^{\frac{1}{2}}} = \overline{\text{dom } H}.$$

Therefore the following statements are equivalent:

$$(2.16) \quad \text{dom } H \text{ closed; } \text{dom } H^{\frac{1}{2}} \text{ closed; } \text{dom } H = \text{dom } H^{\frac{1}{2}}.$$

Let H be a nonnegative selfadjoint relation. Then for each $x > 0$,

$$(2.17) \quad \text{dom } (H + x)^{1/2} = \text{dom } H^{1/2},$$

and, moreover,

$$(2.18) \quad \|(H_s + x)^{1/2}h\|^2 = \|(H^{1/2})_s h\|^2 + x\|h\|^2, \quad h \in \text{dom } H^{1/2}.$$

It is clear that the identity holds for $h \in \text{dom } H$ and since $\text{dom } H$ is a core for $H^{1/2}$ it holds for $h \in \text{dom } H^{1/2}$.

There is a natural ordering for nonnegative selfadjoint relations in a Hilbert space \mathfrak{H} ; it is inspired by the corresponding situation for selfadjoint operators $H_1, H_2 \in \mathbf{B}(\mathfrak{H})$. Two nonnegative selfadjoint relations H_1 and H_2 are said to satisfy the inequality $H_1 \leq H_2$ if

$$(2.19) \quad \text{dom } H_{2s}^{\frac{1}{2}} \subset \text{dom } H_{1s}^{\frac{1}{2}}, \quad \|H_{1s}^{\frac{1}{2}}h\| \leq \|H_{2s}^{\frac{1}{2}}h\|, \quad h \in \text{dom } H_2^{\frac{1}{2}}.$$

It follows from (2.17) and (2.18) that $H_1 \leq H_2$ if and only if $H_1 + x \leq H_2 + x$ for some (and hence for all) $x > 0$.

A sesquilinear form (or form for short) $\mathfrak{t}[\cdot, \cdot]$ in a Hilbert space \mathfrak{H} is a mapping from $\mathfrak{D} \times \mathfrak{D}$ to \mathbb{C} where \mathfrak{D} is a (not necessarily densely defined) linear subspace of \mathfrak{H} , such that it is linear in the first entry and anti-linear in the second entry. The domain $\text{dom } \mathfrak{t}$ is defined by $\text{dom } \mathfrak{t} = \mathfrak{D}$. The corresponding quadratic form $\mathfrak{t}[\cdot]$ is defined by $\mathfrak{t}[\varphi] = \mathfrak{t}[\varphi, \varphi]$, $\varphi \in \text{dom } \mathfrak{t}$. A sesquilinear form \mathfrak{t} is said to be nonnegative if

$$\mathfrak{t}[\varphi] \geq 0, \quad \varphi \in \text{dom } \mathfrak{t}.$$

The semibounded form \mathfrak{t} in \mathfrak{H} is said to be *closed* if for any sequence (φ_n) in $\text{dom } \mathfrak{t}$ one has

$$(2.20) \quad \varphi_n \rightarrow \varphi, \quad \mathfrak{t}[\varphi_n - \varphi_m] \rightarrow 0, \quad \Rightarrow \quad \varphi \in \text{dom } \mathfrak{t}, \quad \mathfrak{t}[\varphi_n - \varphi] \rightarrow 0.$$

The inequality $\mathfrak{t}_1 \leq \mathfrak{t}_2$ for forms \mathfrak{t}_1 and \mathfrak{t}_2 is defined by

$$(2.21) \quad \text{dom } \mathfrak{t}_2 \subset \text{dom } \mathfrak{t}_1, \quad \mathfrak{t}_1[h] \leq \mathfrak{t}_2[h], \quad h \in \text{dom } \mathfrak{t}_2.$$

In particular, $\mathfrak{t}_2 \subset \mathfrak{t}_1$ implies $\mathfrak{t}_1 \leq \mathfrak{t}_2$.

The theory of nonnegative forms can be found in [15]. The representation theorem gives a connection between nonnegative selfadjoint relations and nonnegative closed forms; see [9, 15].

Theorem 2.6 (representation theorem). *Let \mathfrak{t} be a closed nonnegative form in the Hilbert space \mathfrak{H} . Then there exists a nonnegative selfadjoint relation H in \mathfrak{H} such that*

(i) $\text{dom } H \subset \text{dom } \mathfrak{t}$ and

$$(2.22) \quad \mathfrak{t}[\varphi, \psi] = (\varphi', \psi)$$

for every $\{\varphi, \varphi'\} \in H$ and $\psi \in \text{dom } \mathfrak{t}$;

(ii) $\text{dom } H$ is a core for \mathfrak{t} and $\text{mul } H = (\text{dom } \mathfrak{t})^\perp$;

(iii) if $\varphi \in \text{dom } \mathfrak{t}$, $\omega \in \mathfrak{H}$, and

$$(2.23) \quad \mathfrak{t}[\varphi, \psi] = (\omega, \psi)$$

holds for every ψ in a core of \mathfrak{t} , then $\{\varphi, \omega\} \in H$.

The nonnegative selfadjoint relation H is uniquely determined by (i).

The following result is a direct consequence of the representation theorem.

Proposition 2.7. *Let T be a closed linear relation from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} . The nonnegative selfadjoint relation T^*T in the Hilbert space \mathfrak{H} corresponds to the closed nonnegative form*

$$(2.24) \quad \mathfrak{t}[h, k] = (T_s h, T_s k)_{\mathfrak{K}}, \quad h, k \in \text{dom } \mathfrak{t} = \text{dom } T_s = \text{dom } T,$$

and, in particular,

$$(2.25) \quad \mathfrak{t}[h, k] = ((T_s)^\times T_s h, k)_{\mathfrak{K}}, \quad h \in \text{dom } T^*T, \quad k \in \text{dom } \mathfrak{t}.$$

Proof. Since T_s is a closed linear operator, it follows that the form in (2.24) is closed. Clearly, if in (2.24) one assumes that $h \in \text{dom } T^*T = \text{dom } (T_s)^\times T_s$, see (2.12), then (2.25) follows. The result is now obtained from Theorem 2.6. \square

Proposition 2.7 combined with (2.12) in Lemma 2.4 yields the so-called second representation theorem for closed forms.

Corollary 2.8. *Let \mathfrak{t} be a closed nonnegative form in the Hilbert space \mathfrak{H} and let H be the corresponding nonnegative selfadjoint relation H in \mathfrak{H} as in Theorem 2.6. Then*

$$(2.26) \quad \text{dom } \mathfrak{t} = \text{dom } H_s^{\frac{1}{2}} \quad \text{and} \quad \mathfrak{t}[\varphi, \psi] = (H_s^{\frac{1}{2}} \varphi, H_s^{\frac{1}{2}} \psi), \quad \varphi, \psi \in \text{dom } \mathfrak{t}.$$

A subset of $\text{dom } \mathfrak{t} = \text{dom } H_s^{\frac{1}{2}}$ is a core of the form \mathfrak{t} if and only if it is a core of the operator $H_s^{\frac{1}{2}}$. In particular, $\text{dom } H$ is a core of $H^{\frac{1}{2}}$.

As a straightforward consequence of the representation theorem one can state the following result which connects inequalities between nonnegative selfadjoint relations with inequalities between the corresponding nonnegative closed forms.

Corollary 2.9. *Let \mathfrak{t}_1 and \mathfrak{t}_2 be closed nonnegative forms and let H_1 and H_2 be the corresponding nonnegative selfadjoint relations. Then*

$$(2.27) \quad \mathfrak{t}_1 \leq \mathfrak{t}_2 \quad \text{if and only if} \quad H_1 \leq H_2.$$

Corollary 2.10. *Let \mathfrak{H} , \mathfrak{H}_1 , and \mathfrak{H}_2 be Hilbert spaces. Let T_1 be a closed linear relation from \mathfrak{H} into \mathfrak{H}_1 and let T_2 be a closed linear relation from \mathfrak{H} into \mathfrak{H}_2 . Then $T_1^*T_1 \leq T_2^*T_2$ if and only if*

$$\text{dom } T_2 \subset \text{dom } T_1 \quad \text{and} \quad \|(T_1)_s h\|_{\mathfrak{H}_1} \leq \|(T_2)_s h\|_{\mathfrak{H}_2}, \quad h \in \text{dom } T_2.$$

Proof. Let \mathfrak{t}_1 and \mathfrak{t}_2 be the closed nonnegative forms in the Hilbert space \mathfrak{H} induced by $T_1^*T_1$ and $T_2^*T_2$. Hence by Corollary 2.9 one has $T_1^*T_1 \leq T_2^*T_2$ if and only if $\mathfrak{t}_1 \leq \mathfrak{t}_2$. By Proposition 2.7 $\mathfrak{t}_1 \leq \mathfrak{t}_2$ if and only if

$$\text{dom } T_2 \subset \text{dom } T_1, \quad ((T_1)_s h, (T_1)_s h)_{\mathfrak{H}_1} \leq ((T_2)_s h, (T_2)_s h)_{\mathfrak{H}_2}, \quad h \in \text{dom } T_2. \quad \square$$

3. THE LEMMA OF DOUGLAS IN THE CONTEXT OF LINEAR RELATIONS

In this section the lemma of Douglas, see Introduction, will be discussed in the context of linear relations. The first result to be presented is about range inclusion and factorisation. It goes back to D. Popovici and Z. Sebestyén [16], who stated it actually in the context of linear spaces. Some refinements can be found in [17].

Proposition 3.1. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a linear relation from \mathfrak{H}_A to \mathfrak{H} , and let B be a linear relation from \mathfrak{H}_B to \mathfrak{H} . Then $\text{ran } A \subset \text{ran } B$ if and only if there exists a linear relation W from \mathfrak{H}_A to \mathfrak{H}_B such that $A \subset BW$.*

Proof. (\Rightarrow) Let the linear relation W from \mathfrak{H}_A to \mathfrak{H}_B be defined by the product

$$W = B^{-1}A.$$

Let $\{f, f'\} \in A$. Then $f' \in \text{ran } A$, so that $f' \in \text{ran } B$ and there exists $\varphi \in \mathfrak{H}_B$ such that $\{\varphi, f'\} \in B$ or $\{f', \varphi\} \in B^{-1}$. Hence $\{f, \varphi\} \in W$ and $\{f, f'\} \in BW$.

(\Leftarrow) Let $f' \in \text{ran } A$, then for some $f \in \mathfrak{H}_A$ one has $\{f, f'\} \in A$. Hence there is $\varphi \in \mathfrak{H}_B$ such that $\{f, \varphi\} \in W$ and $\{\varphi, f'\} \in B$. This implies that $f' \in \text{ran } B$. \square

For the next result, see [17, Proposition 2]; for completeness a short proof is presented.

Proposition 3.2. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a linear relation from \mathfrak{H}_A to \mathfrak{H} , and let B be a linear relation from \mathfrak{H}_B to \mathfrak{H} . Then there exists a linear relation W from \mathfrak{H}_A to \mathfrak{H}_B such that $A = BW$ if and only if*

$$\text{ran } A \subset \text{ran } B \quad \text{and} \quad \text{mul } B \subset \text{mul } A.$$

Proof. (\Rightarrow) It follows from (2.3) that $\text{mul } B \subset \text{mul } BW = \text{mul } A$ while $\text{ran } A \subset \text{ran } B$ holds by Proposition 3.1.

(\Leftarrow) As in the proof of Proposition 3.1 consider $W = B^{-1}A$ which satisfies $A \subset BW$. In view of (2.4) one can write

$$(3.1) \quad BW = BB^{-1}A = (I_{\text{ran } B} \hat{+} (\{0\} \times \text{mul } B)) A.$$

Since $\text{ran } A \subset \text{ran } B$, it is clear from (3.1) that $\text{dom } BW = \text{dom } A$ and since $\text{mul } B \subset \text{mul } A$ one also concludes from (3.1) that $\text{mul } BW = \text{mul } A$. Therefore, the equality $BW = A$ holds by Corollary 2.2. \square

Observe that if W is a linear relation from \mathfrak{H}_A to \mathfrak{H}_B , then the inclusion $A \subset BW$ shows that

$$\text{dom } A \subset \text{dom } W \quad \text{and} \quad \text{ran } A \subset \text{ran } B.$$

Furthermore, if W is an operator, then the inclusion $A \subset BW$ is equivalent to:

$$\text{dom } A \subset \text{dom } W \quad \text{and} \quad \{Wf, f'\} \in B \quad \text{for all} \quad \{f, f'\} \in A,$$

so that in particular W takes $\text{dom } A$ into $\text{dom } B$. Hence when the relation W is a bounded operator then it may be assumed that $W \in \mathbf{B}(\overline{\text{dom } A}, \overline{\text{dom } B})$. In this case the zero continuation W_c of W to $(\text{dom } A)^\perp$ satisfies $A \subset BW \subset BW_c$ and $\|W_c\| = \|W\|$, so that without loss of generality it may be assumed that $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$.

Lemma 3.3. *Let $A \subset BW$ for some $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$. Then*

$$W^*B^* \subset A^* \quad \text{and} \quad A^{**} \subset B^{**}W.$$

Proof. Clearly $A \subset BW$ implies via (2.8) that

$$W^*B^* \subset (BW)^* \subset A^*.$$

This inclusion combined with $W^* \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ and (2.9) in turn gives rise to

$$A^{**} \subset (W^*B^*)^* = B^{**}W^{**} = B^{**}W. \quad \square$$

The main result in this section concerns factorization and majorization. If A and B are closed linear relations, then the case that $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ can be characterized as follows; see also [3, 6].

Theorem 3.4. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a closed linear relation from \mathfrak{H}_A to \mathfrak{H} , and let B be a closed linear relation from \mathfrak{H}_B to \mathfrak{H} . Then there exists an operator $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ (or equivalently an operator $W \in \mathbf{B}(\overline{\text{dom } A}, \overline{\text{dom } B})$) such that*

$$(3.2) \quad A \subset BW,$$

if and only if there exists $c \geq 0$ such that

$$(3.3) \quad AA^* \leq c^2 BB^*.$$

One can take $\|W\| \leq c$.

Proof. (\Rightarrow) Let $A \subset BW$ with $W \in \mathbf{B}(\overline{\text{dom } A}, \overline{\text{dom } B})$. By considering the zero continuation of W , again denoted by W , it may be assumed that $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$. Then

$$(3.4) \quad W^*B^* \subset A^*,$$

cf. Lemma 3.3. In particular this implies that $\text{dom } B^* \subset \text{dom } A^*$. Now let $\{f, f'\} \in (B^*)_s \subset B^*$. Then it follows from (3.4) that

$$\{f, W^*f'\} \in A^*.$$

Hence there is an element $\chi \in \text{mul } A^*$ such that

$$W^*(B^*)_s f = (A^*)_s f + \chi.$$

Observe that

$$\|(A^*)_s f\|^2 \leq \|(A^*)_s f\|^2 + \|\chi\|^2 = \|W^*(B^*)_s f\|^2 \leq \|W\|^2 \|(B^*)_s f\|^2.$$

Together with $\text{dom } B^* \subset \text{dom } A^*$ this inequality proves (3.3); see Corollary 2.10.

(\Leftarrow) Assume that (3.3) holds, in other words, assume that there exists $c \geq 0$ such that

$$(3.5) \quad \text{dom } B^* \subset \text{dom } A^*, \quad c \|(B^*)_s f\| \geq \|(A^*)_s f\|, \quad f \in \text{dom } B^*.$$

Consider A_s as a densely defined operator from $\overline{\text{dom } A}$ to $(\text{mul } A)^\perp$ and B_s as a densely defined operator from $\overline{\text{dom } B}$ to $(\text{mul } B)^\perp$. Then the assumption (3.5) is equivalent to

$$(3.6) \quad \text{dom } B^* \subset \text{dom } A^*, \quad c \|(B_s)^\times f\| \geq \|(A_s)^\times f\|, \quad f \in \text{dom } B^*,$$

where the adjoints $(A_s)^\times$ and $(B_s)^\times$ are with respect to these smaller spaces; see (2.7). Define the linear relation D by

$$D = \{(B_s)^\times f, (A_s)^\times f\} : f \in \text{dom } B^* \}.$$

Then by (3.6) D is a bounded operator from $\overline{\text{dom } B}$ to $\overline{\text{dom } A}$ with $\|D\| \leq c$. It has a unique extension, again denoted by D , from $\overline{\text{dom } B}$ to $\overline{\text{dom } A}$ with $\|D\| \leq c$, such that

$$D(B_s)^\times \subset (A_s)^\times,$$

or taking adjoints, using (2.9),

$$(3.7) \quad A_s = (A_s)^{\times \times} \subset (D(B_s)^\times)^\times = (B_s)^{\times \times} D^\times = B_s W_0,$$

where $W_0 = D^\times$ is a bounded linear operator from $\overline{\text{dom } A}$ to $\overline{\text{dom } B}$, with $\|W_0\| = \|D\| \leq c$. Observe that the inclusion $\text{dom } B^* \subset \text{dom } A^*$ implies that

$$(3.8) \quad \text{mul } A \subset \text{mul } B.$$

Now let $\{f, f'\} \in A$, so that $f' = A_s f + \varphi$ with $\varphi \in \text{mul } A$. By (3.8) one has $\varphi \in \text{mul } B$. By (3.7) the inclusion $\{f, A_s f\} \in A_s$ implies that

$$\{f, W_0 f\} \in W_0, \quad \{W_0 f, A_s f\} \in B_s,$$

and, hence

$$\{W_0 f, A_s f + \varphi\} \in B.$$

Therefore one concludes that $\{f, f'\} \in BW_0$, i.e., $A \subset BW_0$ holds with $W_0 \in \mathbf{B}(\overline{\text{dom } A}, \overline{\text{dom } B})$. Finally, let W be the zero continuation of W_0 to $(\text{dom } A)^\perp$. Then $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ with $\|W\| = \|W_0\|$ and, moreover, the inclusion $A \subset BW$ is satisfied. \square

In particular, the equivalences $AA^* \leq BB^* \Leftrightarrow W^*B^* \subset A^* \Leftrightarrow A \subset BW$ with $\|W\| \leq 1$ can be found in [3, Proposition 2.2, Remark 2.3]. For densely defined operators A and B the implication $AA^* \leq BB^* \Rightarrow A \subset BW$, $\|W\| \leq 1$, can be found in [6, Theorem 2].

The following two corollaries are variations on the theme of Theorem 3.4.

Corollary 3.5. *Let A and B be closed linear relations as in Theorem 3.4 and, in addition, let $T \in \mathbf{B}(\mathfrak{K}, \mathfrak{H}_A)$ with \mathfrak{K} a Hilbert space. Then*

$$AA^* \leq c^2 BB^* \quad \Rightarrow \quad AT\overline{T^*A^*} \leq c^2 \|T\|^2 BB^*,$$

where $c \geq 0$. In particular,

$$BT\overline{T^*B^*} \leq \|T\|^2 BB^*$$

holds for every $T \in \mathbf{B}(\mathfrak{K}, \mathfrak{H}_A)$.

Proof. Assume that $AA^* \leq c^2 BB^*$, which by Theorem 3.4 is equivalent to the inclusion $A \subset BW$. Therefore it follows that

$$AT \subset BWT.$$

Observe that AT is closed and that WT is bounded. Hence again by Theorem 3.4 one obtains

$$AT(AT)^* \leq \|WT\|^2 BB^*$$

Now observe that (2.9) shows that

$$(AT)^* = ((T^*A^*)^*)^* = \overline{T^*A^*}$$

Hence this leads to

$$AT\overline{T^*A^*} \leq c_T^2 BB^*,$$

where one can take $c_T = \|WT\| \leq c\|T\|$. The last statement follows from the first one with the choices $A = B$ and $c = 1$. \square

Corollary 3.6. *Let A and B be closed linear relations as in Theorem 3.4 and let T be a linear relation from the Hilbert space \mathfrak{H} to the Hilbert space \mathfrak{K} . Then*

$$(3.9) \quad AA^* \leq c^2 BB^* \quad \Rightarrow \quad \overline{TA}(TA)^* \leq c^2 \overline{TB}(TB)^*,$$

where $c \geq 0$. In particular, if $T \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ then

$$AA^* \leq c^2 BB^* \quad \Rightarrow \quad \overline{TAA^*}T^* \leq c^2 \overline{TBB^*}T^*.$$

Proof. Assume that $AA^* \leq c^2 BB^*$. Then by Theorem 3.4 $A \subset BW$ for some $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$ with $\|W\| \leq c$. Hence it follows that

$$(3.10) \quad TA \subset T(BW) = (TB)W \subset \overline{TBW}.$$

Due to (2.9) the following identity holds

$$\overline{TBW} = (W^*(TB)^*)^*,$$

which implies that the relation \overline{TBW} is closed. Therefore one concludes from (3.10) that

$$AA^* \leq c^2 BB^* \Rightarrow \overline{TA} \subset \overline{TBW}.$$

By Theorem 3.4 this implication can be rewritten as the implication stated in (3.9). If $T \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ the last statement is obtained by applying (2.9) to (3.9). \square

The occurrence of the equality $A = BW$ in Theorem 3.4 can be characterized as follows.

Proposition 3.7. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a closed linear relation from \mathfrak{H}_A to \mathfrak{H} , and let B be a closed linear relation from \mathfrak{H}_B to \mathfrak{H} . Then there exists a bounded (not necessarily closed) operator W from $\text{dom } A$ into $\text{dom } B$ such that*

$$(3.11) \quad A = BW,$$

if and only if the following conditions are satisfied

- (i) *the inequality (3.3) holds for some $c \geq 0$;*
- (ii) *$\text{mul } A = \text{mul } B$.*

Proof. (\Rightarrow) If $A = BW$ holds for some bounded operator W from $\text{dom } A$ into $\overline{\text{dom } B}$, then clearly $A \subset BW^{**}$ and here $W^{**} \in \mathbf{B}(\overline{\text{dom } A}, \overline{\text{dom } B})$, since $\text{dom } A \subset \text{dom } W$. Now the inequality (3.3) is obtained from Theorem 3.4. Since W is an operator, one obtains $\text{mul } A = \text{mul } BW = \text{mul } B$; see (2.3).

(\Leftarrow) The inequality (3.3) implies the existence of $W_0 \in \mathbf{B}(\overline{\text{dom } A}, \overline{\text{dom } B})$ such that $A \subset BW_0$ by Theorem 3.4. Then $\text{dom } A \subset \text{dom } W_0$ and the restriction $W := W_0 \upharpoonright \text{dom } A$ is a bounded operator such that $A \subset BW$ and $\text{dom } BW = \text{dom } A$. The second assumption implies that $\text{mul } BW = \text{mul } B = \text{mul } A$ and hence the equality $A = BW$ follows from Corollary 2.2. \square

The following result concerns the alternative formulation of the Douglas lemma which is known in the literature, but now in the context of relations. The domain condition is a sufficient condition.

Proposition 3.8. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a closed linear relation from \mathfrak{H}_A to \mathfrak{H} , and let B be a closed linear relation from \mathfrak{H}_B to \mathfrak{H} . Assume that $\text{dom } A^* = \text{dom } B^*$. Then the following statements are equivalent:*

- (i) *$AA^* \leq c^2 BB^*$ for some $c \geq 0$;*
- (ii) *$AA^* = BMB^*$ for some $0 \leq M \in \mathbf{B}(\mathfrak{H}_B)$ with $\|M\| \leq c^2$.*

Proof. (i) \Rightarrow (ii) By Theorem 3.4 it follows that $A \subset BW$, and that $W^*B^* \subset A^*$. Let Q be the orthogonal projection onto $(\text{mul } A^*)^\perp$. Then clearly

$$QW^*B^* \subset QA^*,$$

where QA^* is an operator. The assumption $\text{dom } A^* = \text{dom } B^*$ implies that actually equality holds

$$QW^*B^* = QA^*.$$

Therefore one obtains via $AA^* = AQA^*$, see Lemma 2.4, that

$$AA^* = AQA^* \subset BWQA^* = BWQW^*B^* = (BWQ)(QW^*B^*),$$

where the relation BWQ is closed and

$$(QW^*B^*)^* = B(QW^*)^* = BWQ.$$

Hence the term $(BWQ)(QW^*B^*)$ is selfadjoint and equality prevails:

$$AA^* = BWQW^*B^* = BMB^* \quad \text{with} \quad M = WQW^*.$$

Note that $\|M\| \leq \|W\|^2 \leq c^2$.

(ii) \Rightarrow (i) Since $M \geq 0$ is bounded one can rewrite (ii) in the form

$$(3.12) \quad AA^* = BMB^* = BM^{1/2}M^{1/2}B^* \subset (BM^{1/2})\overline{M^{1/2}B^*}.$$

Observe that by (2.9)

$$\overline{M^{1/2}B^*} = (M^{1/2}B^*)^{**} = (BM^{1/2})^*.$$

This equality and the fact that $BM^{1/2}$ is closed together show that both sides in (3.12) are selfadjoint; see Lemma 2.4. Thus there is actually equality in (3.12):

$$AA^* = (BM^{1/2})\overline{M^{1/2}B^*}.$$

Now Corollary 3.5 implies that

$$AA^* = BM^{1/2}\overline{M^{1/2}B^*} \leq \|M\|BB^*,$$

so that (3.3) follows with $c^2 = \|M\|$. \square

4. DOMINATION OF LINEAR RELATIONS

The following notions and terminology are strongly influenced by the theory of Lebesgue type decompositions of linear relations and forms, cf. [11], [12], [19]. In fact in these papers the notion of domination is used for (mostly closable) operators. However domination can be defined also in the context of linear relations as follows.

Definition 4.1. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a linear relation from \mathfrak{H} to \mathfrak{H}_A , and let B be a linear relation from \mathfrak{H} to \mathfrak{H}_B . Then B dominates A if there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that*

$$(4.1) \quad ZB \subset A.$$

Note that the inclusion $ZB \subset A$ in (4.1) means that

$$(4.2) \quad \{\{f, Zf'\} : \{f, f'\} \in B\} \subset A.$$

This shows that $\text{dom } B \subset \text{dom } A$ and that $\ker B \subset \ker A$. Furthermore,

$$\text{mul } ZB = Z(\text{mul } B) \subset \text{mul } A.$$

It follows from the definition that Z takes $\text{ran } B$ into $\text{ran } A$; the boundedness implies that Z takes $\overline{\text{ran } B}$ into $\overline{\text{ran } A}$. Hence one can assume that $(\text{ran } B)^\perp \subset \ker Z$, in which case Z is uniquely determined. Domination is transitive: if $Z_1B \subset A$ and $Z_2C \subset B$ then

$$Z_1(Z_2C) \subset Z_1B \subset A,$$

so that $(Z_1Z_2)C \subset A$.

Let A and B be relations in a Hilbert space \mathfrak{H} which satisfy $B \subset A$. Then clearly B dominates A (with $Z = 1$). In particular, since $A \subset A^{**}$, it follows that A dominates A^{**} .

In the particular case when A and B in the above definition are linear operators it is possible to give an equivalent characterization of domination.

Lemma 4.2. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a linear operator from \mathfrak{H} to \mathfrak{H}_A , and let B be a linear operator from \mathfrak{H} to \mathfrak{H}_B . Then B dominates A if and only if there exists $c \geq 0$ such that*

$$(4.3) \quad \text{dom } B \subset \text{dom } A \quad \text{and} \quad \|Af\| \leq c\|Bf\|, \quad f \in \text{dom } B.$$

Proof. Assume that B dominates A . Then (4.1) shows that $\text{dom } B \subset \text{dom } A$ and that for all $f \in \text{dom } B$ one has $ZBf = Af$, which leads to

$$\|Af\| \leq \|Z\|\|Bf\|, \quad f \in \text{dom } B.$$

The desired result follows from this with $c = \|Z\|$.

Conversely, assume that (4.3) holds. Define an operator Z_0 from $\text{ran } B$ to $\text{ran } A$ by $Z_0Bf = Af$, $f \in \text{dom } B$. It follows from (4.3) that the operator Z_0 is well defined and bounded with $\|Z_0\| \leq c$. Thus Z_0 can be continued to a bounded operator from $\overline{\text{ran } B}$ to $\overline{\text{ran } A}$ with the same norm. Let Z be the extension of $\text{clos } Z_0$ obtained by defining Z to be 0 on $(\text{ran } B)^\perp$. Then clearly $Z : \mathfrak{H}_B \rightarrow \mathfrak{H}_A$ is bounded and $ZB \subset A$ holds. \square

A weaker version of Lemma 4.2 with densely defined operators on a Banach space appears in [8, Theorem 2.8]; see also [4, 7].

Lemma 4.3. *Let the relation B dominate the relation A as in (4.1), then*

$$(4.4) \quad A^* \subset B^*Z^*,$$

and, consequently

$$(4.5) \quad ZB^{**} \subset A^{**}.$$

*In other words, B^{**} dominates A^{**} with the same operator Z . In particular, if B dominates A then the following inclusions are valid*

$$\text{dom } B \subset \text{dom } A, \quad \text{ran } A^* \subset \text{ran } B^*, \quad \text{and} \quad \text{dom } B^{**} \subset \text{dom } A^{**}.$$

Proof. It follows from (4.1) and (2.9) that

$$A^* \subset (ZB)^* = B^*Z^*.$$

Now taking adjoints again yields

$$Z^{**}B^{**} \subset (B^*Z^*)^* \subset A^{**},$$

and this proves (4.5). The remaining statement are clear from (4.4) and (4.5). \square

So far domination has been defined for linear relations which are not necessarily closed. Due to Lemma 4.3 domination of closed linear relations can be characterized in terms of majorization.

Theorem 4.4. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a closed linear relation from \mathfrak{H} to \mathfrak{H}_A , and let B be a closed linear relation from \mathfrak{H} to \mathfrak{H}_B . Then there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that*

$$(4.6) \quad ZB \subset A$$

if and only if there exists $c \geq 0$ such that

$$(4.7) \quad A^*A \leq c^2 B^*B.$$

One can take $\|Z\| \leq c$.

Proof. Since A and B are assumed to be closed the inclusions (4.6) and (4.4) are equivalent. Hence the result follows from Theorem 3.4. \square

Proposition 4.5. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a closed linear relation from \mathfrak{H} to \mathfrak{H}_A , and let B be a closed linear relation from \mathfrak{H} to \mathfrak{H}_B . Then there exists an operator $Z \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H}_A)$ such that*

$$(4.8) \quad A = ZB$$

if and only if the following three conditions are satisfied:

- (i) *the inequality (4.7) holds for some $c \geq 0$;*
- (ii) *$\text{dom } A = \text{dom } B$;*
- (iii) *$\dim(\text{mul } A) \leq \dim(\text{mul } B)$.*

Proof. (\Rightarrow) Property (i) follows directly from Theorem 4.4. Since $\text{dom } Z = \mathfrak{H}_B$, the equality (4.8) implies (ii). Finally, it follows from (2.3) and (4.8) that $\text{mul } A = \text{mul } ZB = Z(\text{mul } B)$, i.e. Z maps $\text{mul } B$ onto $\text{mul } A$, and hence (iii) holds.

(\Leftarrow) Decompose A and B via their operator parts:

$$A = A_s \hat{\oplus} A_{\text{mul}}, \quad B = B_s \hat{\oplus} B_{\text{mul}}.$$

By Lemma 2.4 the condition (4.7) is equivalent to $(A_s)^*A_s \leq c^2(B_s)^*B_s$, $c \geq 0$. Now by Theorem 4.4 there exists $Z_0 \in \mathbf{B}(\mathfrak{H}_B \ominus \text{mul } B, \mathfrak{H}_A \ominus \text{mul } A)$ such that

$$Z_0 B_s \subset A_s.$$

By the condition (ii) $\text{dom } A_s = \text{dom } B_s$ and hence, in fact, the equality $Z_0 B_s = A_s$ prevails. Moreover, the condition (iii) guarantees the existence of a surjective operator $Z_m \in \mathbf{B}(\text{mul } B, \text{mul } A)$. Finally, by taking $Z = Z_0 \oplus Z_m$ one gets the desired identity $ZB = A$. \square

Finally note that the result in Proposition 3.8 has a counterpart in the setting of Theorem 4.4. Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let A be a closed linear relation from \mathfrak{H} to \mathfrak{H}_A , and let B be a closed linear relation from \mathfrak{H} to \mathfrak{H}_B . If $\text{dom } A = \text{dom } B$, then the following statements are equivalent:

- (i) $A^*A \leq c^2 B^*B$, $c \geq 0$;
- (ii) $A^*A = B^*MB$ for some $0 \leq M \in \mathbf{B}(\mathfrak{H}_B)$ with $\|M\| \leq c^2$.

5. MAJORIZATION AND DOMINATION

There is a direct connection between the majorization of bounded operators as in the original Douglas lemma and the notion of domination of linear relations as in Definition 4.1.

Lemma 5.1. *Let \mathfrak{H}_A , \mathfrak{H}_B , and \mathfrak{H} be Hilbert spaces, let $A \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H})$, $B \in \mathbf{B}(\mathfrak{H}_B, \mathfrak{H})$, and $W \in \mathbf{B}(\mathfrak{H}_A, \mathfrak{H}_B)$. Then*

$$(5.1) \quad A = BW \iff WA^{-1} \subset B^{-1}.$$

Proof. First observe that if $H \in \mathbf{B}(\mathfrak{H}_1, \mathfrak{H}_2)$, then

$$(5.2) \quad HH^{-1} = I_{\text{ran } H} \subset I_{\mathfrak{H}_2}, \quad H^{-1}H = I_{\mathfrak{H}_1} \hat{+} (\{0\} \times \ker H) \supset I_{\mathfrak{H}_1},$$

as is clear from (2.4) and (2.5).

(\Rightarrow) Assume that $A = BW$. Then by (5.2) it follows that

$$WA^{-1} \subset B^{-1}BWA^{-1} = B^{-1}AA^{-1} \subset B^{-1}.$$

(\Leftarrow) Assume that $WA^{-1} \subset B^{-1}$. Then by (5.2) it follows that

$$BW \subset BWA^{-1}A \subset BB^{-1}A \subset A,$$

so that $BW \subset A$. Actually equality $BW = A$ prevails here, since both BW and A are everywhere defined operators. \square

In other words, the lemma expresses the fact that when A and B are bounded operators, then B majorizes A in the sense of $AA^* \leq \lambda BB^*$ (cf. Lemma 1.1) if and only if the relation A^{-1} dominates the relation B^{-1} in the sense of Definition 4.1

The connection in Lemma 5.1 is useful as it yields a particularly simple proof for the characterization of the ordering of nonnegative selfadjoint relations as in (2.19). For earlier treatments of the ordering, see [5, 9].

Theorem 5.2. *Let H_1 and H_2 be nonnegative selfadjoint relations in a Hilbert space \mathfrak{H} . Then the following statements are equivalent:*

- (i) $H_1 \leq H_2$;
- (ii) $(H_1 + x)^{-1} \geq (H_2 + x)^{-1}$ for some and hence for every $x > 0$;
- (iii) $H_1^{-1} \geq H_2^{-1}$.

Proof. (i) \Leftrightarrow (ii) Recall that $H_1 \leq H_2$ if and only if for some (and hence for all) $x > 0$

$$H_1 + x \leq H_2 + x,$$

and note that for $x > 0$ the inverses $(H_1 + x)^{-1}$ and $(H_2 + x)^{-1}$ belong to $\mathbf{B}(\mathfrak{H})$. By Theorem 4.4 $H_1 + x \leq H_2 + x$ is equivalent to the existence of $Z \in \mathbf{B}(\mathfrak{H})$ such that

$$(5.3) \quad Z(H_2 + x)^{1/2} \subset (H_1 + x)^{1/2}, \quad \|Z\| \leq 1;$$

cf. Corollary 2.10. Now an application of Lemma 5.1 shows that (5.3) is equivalent to

$$(5.4) \quad (H_2 + x)^{-1/2} = (H_1 + x)^{-1/2}Z.$$

Finally, by Lemma 1.1 (or Theorem 3.4) (5.4) is equivalent to

$$(H_2 + x)^{-1/2}(H_2 + x)^{-1/2} \leq (H_1 + x)^{-1/2}(H_1 + x)^{-1/2},$$

since $\|Z\| \leq 1$.

(ii) \Leftrightarrow (iii) Let H be a nonnegative selfadjoint relation. Then clearly also H^{-1} is a nonnegative selfadjoint relation and it is connected to H via

$$(5.5) \quad (H + x)^{-1} = \frac{1}{x} - \frac{1}{x^2} \left(H^{-1} + \frac{1}{x} \right)^{-1},$$

where $x > 0$. Hence for a pair of nonnegative selfadjoint relations H_1 and H_2 one obtains for each $x > 0$:

$$(H_2 + x)^{-1} - (H_1 + x)^{-1} = \frac{1}{x^2} \left[\left(H_1^{-1} + \frac{1}{x} \right)^{-1} - \left(H_2^{-1} + \frac{1}{x} \right)^{-1} \right].$$

Now the equivalence is obtained from (i) \Leftrightarrow (ii). \square

Acknowledgement. The first author is grateful for the support from the Emil Aaltonen Foundation.

REFERENCES

- [1] T. Ando, “Lebesgue-type decomposition of positive operators”, *Acta Sci. Math. (Szeged)*, 38 (1976), 253–260.
- [2] R. Arens, “Operational calculus of linear relations”, *Pacific J. Math.*, 11 (1961), 9–23.
- [3] Yu. Arlinskii and S. Hassi, “ Q -functions and boundary triplets of nonnegative operators”, *Linear Operators and Linear Systems* (volume dedicated to Lev Sakhnovich), (to appear).
- [4] B.A. Barnes, “Majorization, range inclusion, and factorization for bounded linear operators”, *Proc. Amer. Math. Soc.*, 133 (2005), 155–162.
- [5] E.A. Coddington and H.S.V. de Snoo, “Positive selfadjoint extensions of positive symmetric subspaces”, *Math. Z.*, 159 (1978), 203–214.
- [6] R.G. Douglas, “On majorization, factorization, and range inclusion of operators on Hilbert space”, *Proc. Amer. Math. Soc.*, 17 (1966), 413–416.
- [7] M.R. Embry, “Factorization of operators on a Banach space”, *Proc. Amer. Math. Soc.*, 38 (1973), 587–590.
- [8] M. Forough, “Majorization, range inclusion, and factorization for unbounded operators on Banach spaces”, *Linear Alg. Appl.*, 449 (2014), 60–67.
- [9] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, “Form sums of nonnegative selfadjoint operators”, *Acta Math. Hungar.*, 111 (2006), 81–105.
- [10] S. Hassi, Z. Sebestyén, and H.S.V. de Snoo, “On the nonnegativity of operator products”, *Acta Math. Hungar.*, 109 (1–2) (2005), 1–14.
- [11] S. Hassi, Z. Sebestyén, and H.S.V. de Snoo, “Lebesgue type decompositions for nonnegative forms”, *J. Funct. Anal.*, 257 (2009), 3858–3894.
- [12] S. Hassi, Z. Sebestyén, and H.S.V. de Snoo, “Lebesgue type decompositions of unbounded linear operators and relations”, in preparation.
- [13] S. Hassi, Z. Sebestyén, H.S.V. de Snoo, and F.H. Szafraniec, “A canonical decomposition for linear operators and linear relations”, *Acta Math. Hungar.*, 115 (2007), 281–307.
- [14] S. Hassi, H.S.V. de Snoo, and F.H. Szafraniec, “Componentwise and Cartesian decompositions of linear relations”, *Dissertationes Mathematicae* 465, Polish Academy of Sciences, Warszawa, 2009, 59 pp.
- [15] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1980.
- [16] D. Popovici and Z. Sebestyén, “Factorizations of linear relations”, *Adv. Math.*, 233 (2013), 40–55.
- [17] A. Sandovici and Z. Sebestyén, “On operator factorization of linear relations”, *Positivity*, 17 (2013), 1115–1122.
- [18] Z. Sebestyén and Zs. Tarsay, “ T^*T always has a positive selfadjoint extension”, *Acta Math. Hungar.*, 135 (2012), 116–129.
- [19] Z. Sebestyén, Zs. Tarsay, and T. Titkos, “Lebesgue decomposition theorems”, *Acta Sci. Math. (Szeged)*, 79 (2013), 219–233.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF VAASA, P.O. BOX 700, 65101 VAASA, FINLAND

E-mail address: sha@uwasa.fi

JOHANN BERNOULLI INSTITUTE FOR MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF GRONINGEN, P.O. BOX 407, 9700 AK GRONINGEN, NEDERLAND

E-mail address: desnoo@math.rug.nl